Degrees of Recovery and Inclusion in Belief Base Dynamics

Márcio Moretto Ribeiro and **Renata Wassermann**
Department of Computer Science
University of São Paulo, Brasil
{marciomr,renata}@ime.usp.br

Abstract

When contracting a formula from a belief base, two desiderata compete: one wants to avoid including any new belief in the process (inclusion) but may want to be able to recover information that was in the base before the contraction took place (recovery).

The AGM paradigm imposes both constraints on contraction operations. However, for finite belief bases inclusion and recovery cannot be simultaneously satisfied.

In this paper, we examine constructions that weaken the inclusion constraint and retain some form of recovery. We show that depending on what is allowed to be added, we obtain a counterpart to the principle of minimal change, where we add just enough information to allow recovery.

Introduction

Belief Revision (Gärdenfors 1988; Gärdenfors and Rott 1995; Hansson 1999) deals with the problem of accommodating new information into a body of existing beliefs. The new piece of information may be inconsistent with the previous information held by the agent. In this case, he may have to give up some previous beliefs.

The problem has been extensively studied in the literature and most formal proposals derive from what is known as the *AGM paradigm*, due to the initials of the authors of (Alchourrón, Gärdenfors, and Makinson 1985). In the AGM paradigm, three operations of belief change are distinguished: expansion, which is the simple addition of a new belief; contraction, which consists in removing the desired belief; and revision, which consists in adding a new belief in such a way that the resulting set is consistent.

In the AGM paradigm, the beliefs of an agent are represented by a set of formulas closed under logical consequence, a *belief set*. The operation of expansion is obtained by adding the new belief and closing the resulting set under logical consequence. The operations of contraction and revision are not uniquely defined, but restricted by a set of desired axioms (the *rationality postulates*). Several mathematical constructions were proposed that have the property of being equivalent to the set of postulates, in the sense that not only an operation following these constructions satisfies the postulates, but also any operation satisfying the postulates can be obtained from these constructions (Gärdenfors 1988). All of these constructions satisfy the Levi identity, that shows how revision can be obtained from contraction and expansion. Therefore, in this paper we will concentrate on the contraction operation.

The six basic AGM postulates for contraction are listed below:

(K-1) $K - \varphi$ is a belief set (*closure*)
(K-2) $K - \varphi \subseteq K$ (*inclusion*)
(K-3) If $\varphi \notin K$, then $K - \varphi = K$ (*vacuity*)
(K-4) If not $\vdash \varphi$, then $\varphi \notin K - \varphi$ (*success*)
(K-5) $K \subseteq (K - \varphi) + \varphi$ (*recovery*)
(K-6) If $\vdash \varphi \leftrightarrow \psi$, then $K - \varphi = K - \psi$ (*extensionality*)

These postulates are supposed to capture the intuition behind the operation of giving up a belief in a rational way. Postulate **(K-1)** says that the result of contracting a belief set by a formula should again be a belief set. The next postulate assures that in an operation of contraction no new formulas are added to the initial belief set. If the formula to be contracted is not an element of the initial belief set, then by **(K-3)** nothing changes. Postulate **(K-4)** says that unless the sentence to be contracted is logically valid (and hence, an element of every theory), it is not an element of the resulting belief set. The recovery postulate **(K-5)** is the most controversial one (Makinson 1987). It says that a contraction should be recoverable, that is, that the original belief set should be recovered by expanding by the formula that was contracted. The last postulate assures that contraction by logically equivalent sentences produces the same output.

The postulate of *recovery* has been debated since the very beginning of the AGM paradigm. We want to avoid contraction operations that simply discard all the beliefs. Intuitively, when we contract by a formula α, we want to discard a minimal subset of the belief set such that the resulting set does not contain α. This is known as the Principle of Minimal Change. Recovery is one way to try to capture this minimality, but some examples show that the postulate may be too strong:

Example 1 (Hansson 1999): "I have read in a book about Cleopatra that she had both a son and a daughter. My set of beliefs therefore contains both p and q, where p denotes that Cleopatra had a son and q that she had a daughter. I then learn from a knowledgeable friend that the book is in fact a historical novel. After that I contract $p \vee q$ from my set of beliefs, i.e., I do not any longer believe that Cleopatra had a child. Soon after that, however, I learn from a reliable source that Cleopatra had a child. It seems perfectly reasonable for me to then add $p \vee q$ to my set of beliefs without also reintroducing either p or q. This contradicts Recovery."

Operations that satisfy **(K-1)**-**(K-4)** and **(K-6)**, i.e., that do not satisfy recovery, were called *withdrawals* in (Makinson 1987). There are several constructions in the literature that do not satisfy recovery (Makinson 1987; Fermé 2001; Rott and Pagnucco 1999), but for belief sets they all present some other undesirable property.

The postulates of *success* and *inclusion* are sometimes seen as the minimal requirements for a contraction operation. Together they state that the desired belief is removed and nothing else new is included in the belief set. (Booth et al. 2005) calls *retraction* an operation that satisfies **(K-1)**, **(K-3)**, **(K-4)** and **(K-6)** plus $K - \perp = K$ (*failure*). A retraction does not satisfy recovery or inclusion. The idea is that the contraction of a formula can "liberate" other beliefs that were abandoned because of that formula.

An alternative representation to belief sets is the use of belief bases, i.e., sets of formulas that are not necessarily closed under logical consequence. Besides being more expressive (as we can always derive the corresponding belief set taking the closure of a belief base), belief bases have clear advantages from the computational point of view. For belief bases, there are constructions that seem reasonable and do not satisfy recovery, as we will show later. The use of belief bases instead of logically closed sets brings with it the possibility of distinguishing between different syntactical representations of the same information. The belief bases $B_1 = \{p, q\}$, $B_2 = \{p \wedge q\}$, and $B_3 = \{p, p \rightarrow q\}$ are all different, although they imply exactly the same formulas according to classical logic. We can look at this possibility as an advantage in terms of expressivity. If we do not want to distinguish between the three cases, we only have to look to the formulas in the logical closure of the bases.

In this paper we will discuss the role and adequacy of the inclusion postulate and its relation to recovery. If we move from closed belief sets to belief bases, the postulate may be too restrictive, as the following examples show:

Example 2: Suppose we have a belief base containing $p \wedge q$, which stands for the fact that Cleopatra had a son (p) and a that she had a daughter (q). If we want to contract the base by p, i.e., we want to remove the belief that she had a son, we have to give up the whole conjunction, and since the formula q is not included in the base, we give up the belief that Cleopatra had a daughter too.

Example 3: Suppose I believe that penguins are birds ($p \rightarrow b$) and that birds fly ($b \rightarrow f$). Now I want to contract my belief that penguins fly ($p \rightarrow f$). We may want to end with the belief that all birds except for penguins fly ($(b \wedge \neg p) \rightarrow f$). This is also not allowed if the inclusion postulate holds.

The third example is a typical case of what (Maranhao 2001) calls Refinement, where there is a rule that needs to be weakened in order to accommodate exceptions.

We would like to have a construction that allows us to keep parts of formulas being removed. In this paper, we will present some ideas on such constructions. We will present two constructions for contraction without inclusion: one which is based on the idea of first expanding the base and then applying partial meet contraction (Alchourrón, Gärdenfors, and Makinson 1985) to it and one where we first apply partial meet contraction to the base and then expand the result.

It is important to note that even if we are questioning the adequacy of inclusion, we are concerned about it with respect to belief bases. We are not willing to allow completely new information to be added during contraction (as is the case of retraction), but only to allow the addition of information that was previously derivable. If we look at the closure of the belief base, we are not adding, but just retaining information.

The rest of the paper is organized as follows: in the next section, we introduce partial meet contraction of belief bases and its properties. Then we give an example of contraction without inclusion and propose a more general construction. We show that this construction can be instantiated and give rise to operations with different properties. We also propose a second construction together with a new rationality postulate and show how to introduce the idea of degrees of recovery.

Throughout the paper we use lower case letters to denote atoms, Greek lower case for formulas and upper case letters to denote set of formulas. We consider Cn to be the classical consequence operator and $A \vdash \alpha$ if and only if $\alpha \in Cn(A)$.

Contraction of Belief Bases

In this section, we will present postulates and a construction for contraction of belief bases. We will then discuss the properties of the operation.

The first construction for contraction that was proposed and proved to be equivalent to the six AGM postulates was *Partial Meet Contraction* (Alchourrón, Gärdenfors, and Makinson 1985). The operation is based on the idea of selecting maximal subsets of a belief set that do not imply the formula to be contracted.

Given a belief set K and a formula α, the *remainder* of K and α, denoted by $K \perp \alpha$ is the set of maximal subsets of K that do not imply α.

Definition 1 *(Alchourrón and Makinson 1982) Let X be a set of formulas and α a formula. For any set Y, $Y \in X \perp \alpha$ if and only if:*

- $Y \subseteq X$
- $Y \nvdash \alpha$
- *For all Y' such that $Y \subset Y' \subseteq X$, $Y' \vdash \alpha$.*

The operation of contraction is based on selecting the best sets in $K \perp \alpha$ and taking their intersection. The choice is encoded into a selection function:

Definition 2 *(Alchourrón, Gärdenfors, and Makinson 1985) A* **selection function** *for K is a function γ such that:*

- *If $K \perp \alpha \neq \emptyset$, then $\emptyset \neq \gamma(K \perp \alpha) \subseteq K \perp \alpha$.*
- *Otherwise, $\gamma(K \perp \alpha) = \{K\}$.*

Definition 3 *(Alchourrón, Gärdenfors, and Makinson 1985) The* **partial meet contraction operator** *on K based on a selection function γ is the operator $-_\gamma$ such that for all sentences α:*

$K -_\gamma \alpha = \bigcap \gamma(K \perp \alpha)$.

The operation of partial meet contraction was proven to be completely axiomatized by the set of postulates **(K-1)**-**(K-6)**:

Theorem 4 *(Alchourrón, Gärdenfors, and Makinson 1985) An operation $-$ is a partial meet contraction if and only if it satisfies postulates* **(K-1)**-**(K-6)**.

The same construction can be used for belief bases. It is easy to see that for belief bases, $-_\gamma$ satisfies **(K-2)**, **(K-4)**, **(K-6)** and a stronger version of **(K-3)** that we call *logical vacuity*: If $\alpha \notin Cn(B)$, then $B - \alpha = B$. To see that it does not satisfy **(K-5)**, we can look at the Cleopatra example:

Example 1 Revisited: Let $B = \{p, q\}$. Then the remainder of B and $p \vee q$ is $B \perp (p \vee q) = \{\emptyset\}$. Hence, $\gamma(B \perp (p \vee q)) = \{\emptyset\}$ and the contraction is given by $B -_\gamma (p \vee q) = \emptyset$.

But adding $p \vee q$ again does not necessarily bring all the information back: $B \not\subseteq B -_\gamma (p \vee q) + (p \vee q) = \{p \vee q\}$.

Given a contraction operator on belief bases, we can define a contraction operator on belief sets generated from it by taking the closure of the result. If $-$ is a contraction on belief bases, then it generates an operator $-'$ such that $Cn(B) -' \alpha = Cn(B - \alpha)$. If $-_\gamma$ is a partial meet base contraction, the example shows that even the operation on belief sets generated from it does not satisfy recovery, i.e.,

$Cn(B) \not\subseteq Cn(B -_\gamma (p \vee q)) + (p \vee q) = Cn(\{p \vee q\})$.

Actually, Hansson (Hansson 1999) has shown that under very general conditions, any base-generated contraction operation fails to satisfy recovery.

Hansson has proposed an alternative axiomatization for partial meet base contraction and proven that it is equivalent to the construction:

Theorem 5 *(Hansson 1992) An operator $-$ is an operator of partial meet base contraction on B if and only if:*

- *If $\alpha \notin Cn(\emptyset)$, then $\alpha \notin Cn(B - \alpha)$ (success)*
- *$B - \alpha \subseteq B$ (inclusion)*
- *If $\beta \in B \setminus (B - \alpha)$, then there is some B' such that $B - \alpha \subseteq B' \subseteq B$, $\alpha \notin Cn(B')$ and $\alpha \in Cn(B' \cup \{\beta\})$ (relevance)*
- *If for all subsets B' of B, $\alpha \in Cn(B')$ if and only if $\beta \in Cn(B')$, then $B - \alpha = B - \beta$ (uniformity)*

The inclusion and success postulates are the same as **(K-2)** and **(K-4)**. The closure postulate (**(K-1)**) does not apply to belief bases. The last postulate (uniformity) is a stronger version of extensionality. Instead of recovery, Hansson suggested the relevance postulate in order to capture the idea of minimal change. In a contraction by α, the only formulas given up are those that somehow contribute to the derivation of α.

As we have seen, adapting the traditional AGM construction of partial meet contraction to belief bases results in an operation that satisfies inclusion and success, but not recovery. In the next section, we will see how weakening inclusion can bring back some sort of recovery.

Contraction without Inclusion

We have seen in the Introduction that even the inclusion postulate may be too strong when talking about belief bases. We would like sometimes to retain parts of the beliefs that are being given up, which means replacing them by some of their consequences. The second Cleopatra example (Example 2) shows that we may want to have $\{p \wedge q\} - p = \{q\}$, i.e., instead of giving up the conjunction, replace it by one of its conjuncts.

Giving up inclusion all together may bring too much freedom for the allowed constructions: we could end up adding just any formula that did not threaten success, such as having $\{p\} - p = \{q\}$. The idea is to weaken the postulate so as to allow the addition of some kinds of formulas. In this section and the next one, we will explore some possibilities. The constructions that we propose satisfy *logical inclusion*, a weaker version of inclusion proposed in (Hansson 1989):

(logical inclusion) $Cn(B - \alpha) \subseteq Cn(B)$

Note that if Cn is Tarskian, logical inclusion is equivalent to $B - \alpha \subseteq Cn(B)$. We will use any of the two forms in this paper. Hansson called an operation satisfying success and logical inclusion a *pseudo-contraction*.

Following the same line, we may think of a weaker version of recovery. Recall that recovery for belief bases would state that $B \subseteq (B - \alpha) \cup \{\alpha\}$. We can relax the postulate requiring only that the original base is contained in the closure of the result of first contracting and then expanding by the same formula:

(logical recovery) $B \subseteq Cn(B - \alpha + \alpha)$

It is easy to see that logical recovery is indeed weaker than recovery. Recovery implies logical recovery, since Cn is Tarskian. If we look again at Example 2, we can think of an operation that satisfies logical recovery but does not satisfy recovery.

Example 2 Revisited: Let $B = \{p \wedge q\}$, and suppose we want to give up the belief in p and retain the belief in q, i.e., we want that the result of removing p from B is $B - p = \{q\}$. If expansion is defined in the usual way, as the simple union of a base and a formula, than $B \not\subseteq (B - p) + p = \{p, q\}$, but $B \subseteq Cn(\{p, q\})$.

Nebel has proposed a construction for pseudo-contraction that satisfies logical recovery. The idea is to add to partial meet contraction some consequences of the formulas of the original belief base so that they allow for the recovery of the contracted sentences:

Definition 6 *(Nebel 1989) Let B be a finite belief base, α a formula and γ a selection function.*

$$B{-}\alpha = \begin{cases} B & \text{if } \alpha \in Cn(\emptyset) \\ \bigcap \gamma(B \perp \alpha) \cup \{\alpha \to \beta | \beta \in B\} & \text{otherwise} \end{cases}$$

It is easy to see that not all formulas of the form $\alpha \to \beta$ have to be added, we can restrict ourselves to those β such that $\beta \in B \setminus \bigcap \gamma(B \perp \alpha)$ and still retain logical recovery:

Proposition 7 *Let B be a finite belief base, α a formula and γ a selection function and define the contraction $B - \alpha$ as:*

$$B{-}\alpha = \begin{cases} B & \text{if } \alpha \in Cn(\emptyset) \\ \bigcap \gamma(B \perp \alpha) \cup \{\alpha \to \beta | \beta \in B \setminus \bigcap \gamma(B \perp \alpha)\} \\ & \text{otherwise} \end{cases}$$

Then $-$ satisfies logical inclusion, logical vacuity, success, logical recovery and extensionality.

Proof: Logical inclusion, logical vacuity, logical recovery and extensionality follow directly from the construction. To see that success is satisfied, suppose that $\alpha \notin Cn(\emptyset)$ and $\alpha \in Cn(B{-}\alpha)$. Then there are $\beta_1, \beta_2, ..., \beta_n$ in $B \setminus \bigcap \gamma(B \perp \alpha)$ such that $\bigcap \gamma(B \perp \alpha) \cup \{\alpha \to \beta_1, \alpha \to \beta_2, ..., \alpha \to \beta_n\} \vdash \alpha$. Using the deduction theorem and the fact that $(\alpha \to \beta) \to \alpha$ is equivalent to α, we have that $\bigcap \gamma(B \perp \alpha) \vdash \alpha$, which we know cannot be the case (since this is the traditional partial meet construction). □

In a sense, Nebel's construction seems to have been coined to satisfy recovery. There is no other intuition about why exactly the implications of the form $\{\alpha \to \beta | \beta \in B\}$ should be added.

If we go back to our second Cleopatra example (Example 2), Nebel's pseudo-contraction (or the more economic form of it) would make $\{p \wedge q\} - p = \{p \to p \wedge q\}$, which means that if we do not believe anymore that Cleopatra had a son, we do not know anything about her having a daughter.

We will propose here a slightly different construction. Suppose that we have a partial meet contraction for a belief base B, with an associated selection function γ. Instead of taking the result of the contraction and then adding some formulas, we will first expand the belief base and then perform a partial meet contraction.

Let Cn^* denote an operation that generates some consequences of a set of formulas, i.e., $Cn^*(A) \subseteq Cn(A)$. As an example, we could have $Cn^*(A) = \{\alpha \vee \beta | \alpha \in A\}$. Given a selection function γ for a set B, we can extend γ to a selection function for a superset of B. Such extension is usually not unique.

Definition 8 *Let γ be a selection function for B and let B^* contain B. An extension of γ to B^* is a selection function γ^* such that for every $Y \in \gamma^*(B^* \perp \alpha)$ there is an $X \in \gamma(B \perp \alpha)$ such that $X \subseteq Y$.*

Observation 9 *Let γ be a selection function for B and let B^* contain B. If γ^* is an extension of γ to B^*, then for any $X \in \gamma(B \perp \alpha)$ there is a $Y \in \gamma^*(B^* \perp \alpha)$ such that $X \subseteq Y$.*

We define the general partial meet pseudo-contraction by first expanding B by a set containing some of the consequences of the formulas that would be given up in a partial meet contraction, and then applying partial meet contraction on the expanded base. The set of formulas that would be given up is given by $\{\beta | \beta \in B \setminus \bigcap \gamma(B \perp \alpha)\}$ and the consequences that will be used are selected by Cn^*. We call B^* the result of expanding B with the selected consequences.

Definition 10 *Let B be a finite belief base, α a formula and γ a selection function for B. The general partial meet pseudo-contraction $B - \alpha$ is given by:*

$$B - \alpha = \begin{cases} B & \text{if } \alpha \in Cn(\emptyset) \\ \bigcap \gamma^*(B^* \perp \alpha) & \text{otherwise} \end{cases}$$

where $B^* = B \cup Cn^*(\{\beta | \beta \in B \setminus \bigcap \gamma(B \perp \alpha)\})$ and γ^* is an extension of γ.

Proposition 11 *The pseudo-contraction operation defined as above satisfies logical inclusion, logical vacuity, success, and extensionality.*

Proof: Directly from the definition and the observation that $\bigcap \gamma(B \perp \alpha)$ satisfies extensionality, thus equivalent formulas generate the same B^*. □

The proposition shows that general pseudo-contraction satisfies some form of inclusion, vacuity, success and extensionality, four of the five relevant AGM postulates (closure is not applicable to belief bases). Moreover, if we consider the operation on belief sets generated from general partial meet pseudo-contraction, it satisfies closure, inclusion, vacuity, success, and extensionality. Whether this construction satisfies logical recovery or not depends on the Cn^* used. It is easy to see that the restricted form of Nebel's pseudo-contraction can be obtained if we take $Cn^*(A) = \{\alpha \to \beta | \beta \in A\}$. In this particular case, we do have logical recovery. And the operation on belief sets generated from it satisfies recovery.

We can think of different definitions for Cn^* depending on the intuitions. For example, if we look once more to the Cleopatra example, we may want to allow the addition of only those consequences which are subformulas of the formulas removed. The pseudo-contraction operation using this definition of Cn^* does not satisfy logical recovery, as can be seen from the following example:

Example 4: Let $B = \{p, p \to q\}$. Then $B \perp q = \{\{p\}, \{p \to q\}\}$. Suppose that $\gamma(B \perp q) = \{\{p \to q\}\}$. Since the only consequence of $p \to q$ that is a subformula of it is the whole formula, we have $B^* = B$. And since γ^* extends γ, we must have $\gamma^*(B \perp q) = \{\{p \to q\}\}$. Hence $B - q = \{p \to q\}$ and $B \not\subseteq Cn(B - q + q)$.

The operation does not even satisfy the weaker postulate of relevance, as inclusion may not hold and thus we may not have a set B' such that $B{-}\alpha \subseteq B' \subseteq B$. It does however satisfy a still weaker version of relevance, introduced by Hansson in (Hannson 1991):

(core-retainment) If $\beta \in B \setminus (B{-}\alpha)$, then there is some B' such that $B' \subseteq B$, $\alpha \notin Cn(B')$ and $\alpha \in Cn(B' \cup \{\beta\})$

Core-retainment, as relevance, states that if a belief is given up in a contraction, then it was relevant for the derivation of the formula contracted. But unlike relevance, it does not require inclusion. In fact, core-retainment is satisfied for any general partial meet pseudo-contraction, regardless of the particular Cn^* used (we do not even need to have $Cn^*(A) \subseteq Cn(A)$):

Proposition 12 *For any operator Cn^*, general partial meet pseudo-contraction satisfies core-retainment.*

Proof: Let $\beta \in B \setminus (B{-}\alpha)$. Then, there is $X \in \gamma^*(B^*\perp\alpha)$ such that $\beta \notin X$. As γ^* extends γ, there must be $Y \in \gamma(B\perp\alpha)$ such that $Y \subseteq X$. Then take $B' = Y$. $\square$

Another way to relax relevance is to follow what was done with inclusion and recovery. Instead of requiring inclusion, we can require logical inclusion:

(logical relevance) If $\beta \in B \setminus (B{-}\alpha)$, then there is some B' such that $B{-}\alpha \subseteq B' \subseteq Cn(B)$, $\alpha \notin Cn(B')$ and $\alpha \in Cn(B' \cup \{\beta\})$.

Logical relevance is satisfied by any general partial meet pseudo-contraction whenever Cn^* selects only classical consequences of the set, i.e., $Cn^*(A) \subseteq Cn(A)$:

Proposition 13 *If for every set A, $Cn^*(A) \subseteq Cn(A)$, then general partial meet pseudo-contraction satisfies logical relevance.*

Proof: Let $\beta \in B \setminus (B{-}\alpha)$. Then, there is $X \in \gamma^*(B^*\perp\alpha)$ such that $\beta \notin X$. Since $B^* \subseteq Cn(B)$, we can make $B' = X$. $\square$

Propositions 13 and 11 show that if Cn^* is such that for every set A, $Cn^*(A) \subseteq Cn(A)$, then general partial meet pseudo-contraction satisfies logical inclusion, logical vacuity, success, logical relevance and extensionality. Moreover, the operation on belief sets generated from it satisfies closure, inclusion, vacuity, success, relevance and extensionality. (Fuhrmann and Hansson 1994) has shown that for belief sets, if an operation satisfies relevance, then it satisfies recovery. We have then the following corollary:

Corollary 14 *A belief set contraction generated from general partial meet pseudo-contraction where for every set A, $Cn^*(A) \subseteq Cn(A)$, satisfies the six AGM postulates for contraction.*

Minimal Additions

In the previous section, we have presented a general construction for contraction without inclusion which was based on the idea of first expanding the base and then applying partial meet to it. In this section, we follow a different approach, closer to Nebel's proposal: we first apply partial meet contraction to the base and then expand the result. In the end of this section we show the relation between both approaches.

Recall that Nebel's construction (Definition 6) adds the set $\{\alpha \rightarrow \beta | \beta \in B\}$ to the result of the partial meet contraction in order to obtain logical recovery. We have shown in Proposition 7 that it suffices to add $\{\alpha \rightarrow \beta | \beta \in B \setminus \bigcap \gamma(B\perp\alpha)\}$. This means adding an implication for each formula that was given up in the contraction. Can we do with less than that? That is, are there cases in which we can add less than that and still retain logical recovery? The answer is clearly yes: consider a base containing $\{p \wedge \neg r, p \wedge \neg r \wedge q\}$ and a contraction by p. Both formulas of the base are given up, but in order to be logically recoverable, we only need to add $p \rightarrow p \wedge \neg r \wedge q$.

We would like to ensure that only formulas really needed for logical recovery are added in the contraction operation. The following postulate is one option:

(core-addition) If $\beta \in (B - \alpha) \setminus B$, then there exist $\beta' \in B \setminus (B - \alpha)$ and $B' \subseteq B - \alpha$ such that $\alpha \rightarrow \beta' \notin Cn(B')$ but $\alpha \rightarrow \beta' \in Cn(B' \cup \{\beta\})$.

This postulate assures that if a new formula is added when performing a contraction, it is needed in order for some belief that was given up to be recoverable. Core-addition provides a counterpart to core-retainment with respect to minimal change: while core-retainment prevents unnecessary loss of beliefs, core-addition avoids unnecessary addition of new beliefs. In particular, the examples 2 and 3 of the introduction do not satisfy inclusion, but they satisfy logical inclusion and core-addition. In the example 2 we can add q to the base because together with p it implies $p \wedge q$. Likewise, in the example 3 we can add $b \wedge \neg p \rightarrow f$ because it helps to recover $b \rightarrow f$.

We are now left with the issue of finding constructions that satisfy core-addition (together with logical inclusion, logical recovery and success). The following definition provides such a construction, using the idea of a minimal set that together with the partial meet contraction recovers the base but does not imply the sentence being contracted:

Definition 15 *Let B be a belief base, α a formula and γ a selection function for B. Let $\Delta(B, \alpha, \gamma)$ be a minimal subset of $Cn(B)$ such that:*

- $\alpha \rightarrow \beta \in Cn(\bigcap \gamma(B\perp\alpha) \cup \Delta(B, \alpha, \gamma)$ *for all* $\beta \in B \setminus (\bigcap \gamma(B\perp\alpha))$
- *For all $X \in B\perp\alpha$, we have $\alpha \notin Cn(X \cup \Delta(B, \alpha, \gamma))$*

We define the Δ-partial-meet pseudo-contraction of B by α as $B - \alpha = \bigcap \gamma(B\perp\alpha) \cup \Delta(B, \alpha, \gamma)$

Given B, α and γ, $\Delta(B, \alpha, \gamma)$ chooses one of the minimal sets that satisfy the two properties, as there may be more than one. We know that at least one such set exists, since the set used in Nebel's construction $\{\alpha \rightarrow \beta | \beta \in B\}$ satisfies the two properties and hence there must be a minimal subset of it that satisfies the properties.

Proposition 16 *Δ-Partial-meet pseudo-contraction satisfies success, logical inclusion, logical recovery, logical vacuity, and core-addition.*

Note that the construction does not always satisfy extensionality, since Δ depends on the particular formula being contracted. If Δ is such that for logically equivalent formulas α and β we always have $\Delta(B, \alpha, \gamma) = \Delta(B, \beta, \gamma)$ then Δ-partial-meet pseudo-contraction satisfies extensionality.

In this case, the operation on belief sets generated from Δ-partial-meet pseudo-contraction satisfies the six AGM postulates for contraction.

We will now show that this construction is a special case of general partial meet pseudo-contraction (Definition 10). Recall that general partial meet pseudo-contraction was defined as $\bigcap \gamma^*(B^* \perp \alpha)$, where $B^* = B \cup Cn^*(B \setminus \bigcap \gamma(B \perp \alpha))$ and γ^* is an extension of γ. The definition of Cn^* was allowed to vary, and only for some definitions we had logical recovery. Let us now consider the case where $Cn^*(B \setminus \bigcap \gamma(B \perp \alpha)) = \Delta$. We will show that $\bigcap \gamma^*(B^* \perp \alpha) = \bigcap \gamma(B \perp \alpha) \cup \Delta$, i.e., that the two operations coincide.

Proposition 17 *Let Δ be a minimal subset of $Cn(B)$ such that:*

- $\alpha \rightarrow \beta \in Cn(\bigcap \gamma(B \perp \alpha) \cup \Delta)$ *for all* $\beta \in B \setminus (\bigcap \gamma(B \perp \alpha))$
- *For all $X \in B \perp \alpha$, we have $\alpha \notin Cn(X \cup \Delta)$*

and let $Cn^*(B \setminus \bigcap \gamma(B \perp \alpha)) = \Delta$. Then $\bigcap \gamma^*(B^* \perp \alpha) = \bigcap \gamma(B \perp \alpha) \cup \Delta$.

Proof: With $Cn^*(B \setminus \bigcap \gamma(B \perp \alpha)) = \Delta$ we have that $B^* = B \cup \Delta$. From the definition of the extension γ^* we know that for every element Y of $\gamma^*((B \cup \Delta) \perp \alpha)$ there is an $X \in \gamma(B \perp \alpha)$ such that $X \subseteq Y$. From the definition of remainders, since X is a maximal subset of B that does not imply α, and since by the definition of Δ we know that $X \cup \Delta$ does not imply α, Y must be $X \cup \Delta$. This means that $\Delta \subseteq Y$ for all $Y \in \gamma^*((B \cup \Delta) \perp \alpha)$ and hence, $\Delta \subseteq \bigcap \gamma^*((B \cup \Delta) \perp \alpha)$. And again by the definition of γ^*, $\bigcap \gamma^*((B \cup \Delta) \perp \alpha) = \bigcap \gamma(B \perp \alpha) \cup \Delta$. □

This proposition shows that Δ-partial-meet pseudo-contraction is a special case of general partial-meet pseudo-contraction that satisfies logical recovery.

Degrees of recovery

In the construction defined above, we have added enough formulas to the belief base in order to be able to recover all the information that was given up during contraction. Δ-Partial-meet pseudo contraction satisfies logical recovery and is minimal in the sense captured by the core-addition postulate.

As exemplified in the introduction, recovery, as well as the logical recovery postulate can not always be accepted. In some cases, however, we may be interested in partial recovery, i.e., in making sure that a subset of the previous beliefs can be recovered. The core-addition postulate tries to avoid unnecessary addition of formulas to the belief base. The additions are minimal in the sense that if some formula is added to the base it is because it helps to recover some previously removed formula.

Δ-Partial-meet pseudo contraction satisfies core-addition as well as logical recovery. However it is possible to define a construction for contraction that satisfies core-addition, but does not necessarily satisfy logical recovery. For this purpose we need to define a function that chooses the formulas that we want to keep recoverable. In this sense, we can talk about different degrees of recovery. This function must return some elements of $B \setminus \bigcap \gamma(B \perp \alpha)$. Formally, given a set of formulas, let f choose a subset of it (for any set X, $f(X) \subseteq X$). We can easily adapt Definition 15 in order to have partial recovery controlled by f, where Δ_f then is the minimal set that allows recovery of the formulas chosen by f:

Definition 18 *Let f be a function as defined above, B a belief base, α a formula and γ a selection function for B. Let $\Delta_f(B, \alpha, \gamma)$ be a minimal subset of $Cn(B)$ such that:*

- $\alpha \rightarrow \beta \in Cn(\bigcap \gamma(B \perp \alpha) \cup \Delta_f(B, \alpha, \gamma))$ *for all* $\beta \in f(B \setminus (\bigcap \gamma(B \perp \alpha)))$
- *For all $X \in B \perp \alpha$, we have $\alpha \notin Cn(X \cup \Delta_f(B, \alpha, \gamma))$.*

We define the Δ_f-partial-meet pseudo-contraction of B by α as $B - \alpha = \bigcap \gamma(B \perp \alpha) \cup \Delta_f(B, \alpha, \gamma)$.

This construction makes clear the relation between the different degrees of recovery and inclusion. At one extreme, if f selects the whole set, we have logical recovery at the price of adding new formulas to the contracted base. At the other extreme, if f selects the empty set, the contraction operation satisfies inclusion, i.e., no new formulas are added. Between these two extremes there is a whole universe of contractions that recover parts of the base, each of which satisfies success, logical inclusion and core-addition.

The adapted construction does not satisfy logical recovery in general, but inherits relevance (and thus, also core-retainment) from the usual partial-meet contraction included in it.

And similarly to what we did to Δ-partial-meet pseudo contraction, we can show that:

Proposition 19 *If $Cn^*(B \setminus \bigcap \gamma(B \perp \alpha)) = \Delta_f(B, \alpha, \gamma)$, then $\bigcap \gamma^*(B^* \perp \alpha) = \bigcap \gamma(B \perp \alpha) \cup \Delta_f(B, \alpha, \gamma)$*

This proposition shows that if Cn^* is such that it chooses the same elements as Δ_f, then Δ_f-partial-meet pseudo-contraction is a special case of general partial meet pseudo-contraction.

Conclusions and Future Work

In this paper, we have discussed how weakening the inclusion requirement can provide some sort of recovery for operations on belief bases.

We have proposed two general constructions based on partial meet that allow some consequences of the original belief base to be added during contraction. We have shown that Nebel's construction (Nebel 1989) is a special case of our proposals. We have then shown that our proposal allows for operations that satisfy recovery (like Nebel's), but also operations that only satisfy weaker versions of it.

We are left with a whole spectrum of operations varying from those where any formula can be added (i.e., no inclusion postulate) to those where nothing can be added. In between, when we use a weaker notion of inclusion, there are constructions satisfying recovery in different degrees.

Future work includes studying other possible construction and proving representation results for them. We also plan to explore the impact of different choices of Cn^*, γ^* and $\Delta(B, \alpha, \gamma)$.

Definition 15 has connections to abduction that we plan to investigate. An abduction problem consists in, given a set A and a formula α, finding a set X such that $A \cup X \vdash \alpha$ and $A \cup X \not\vdash \bot$. In our case, we have the set $(B - \alpha) + \alpha$ and we are looking for a set Δ so that $((B - \alpha) + \alpha) \cup \Delta$ implies all formulas of B.

Acknowledgments: The first author is supported by FAPESP (grant 2006/53028-6) and the second author is partially supported by CNPq (grant 302514/2007-4). This work was developed as part of FAPESP project 2004/14107-2.

References

Alchourrón, C., and Makinson, D. 1982. On the logic of theory change: contraction functions and their associated revision functions. *Theoria* 48:14–37.

Alchourrón, C.; Gärdenfors, P.; and Makinson, D. 1985. On the logic of theory change. *Journal of Symbolic Logic* 50(2):510–530.

Booth, R.; Chopra, S.; Ghose, A.; and Meyer, T. 2005. Belief liberation (and retraction). *Studia Logica* 79:47–72.

Fermé, E. 2001. Five faces of recovery. In Rott, H., and Williams, M.-A., eds., *Frontiers in Belief Revision*. Kluwer.

Fuhrmann, A., and Hansson, S. O. 1994. A survey of multiple contraction. *Journal of Logic, Language and Information* 3(1):39–74.

Gärdenfors, P., and Rott, H. 1995. Belief revision. In *Handbook of Logic in Artificial Intelligence and Logic Programming*, volume IV. Oxford University Press. chapter 4.2.

Gärdenfors, P. 1988. *Knowledge in Flux - Modeling the Dynamics of Epistemic States*. MIT Press.

Hannson, S. O. 1991. Belief contraction without recovery. *Studia Logica* 50(2):251–260.

Hansson, S. O. 1989. New operators for theory change. *Theoria* 55:114–132.

Hansson, S. O. 1992. Reversing the Levi identity. *Journal of Philosophical Logic* 22:637–639.

Hansson, S. O. 1999. *A Textbook of Belief Dynamics*. Kluwer Academic Press.

Makinson, D. 1987. On the status of the postulate of recovery in the logic of theory change. *Journal of Philosophical Logic* 16:383–394.

Maranhao, J. 2001. Refinement: a tool to deal with inconsistencies. In *ICAIL '01: Proceedings of the 8th international conference on Artificial intelligence and law*, 52–60. ACM.

Nebel, B. 1989. A knowledge level analysis of belief revision. In Brachman, R.; Levesque, H.; and Reiter, R., eds., *First International Conference on Principles of Knowledge Representation and Reasoning - KR'89*, 301–311. Toronto, ON: Morgan Kaufmann.

Rott, H., and Pagnucco, M. 1999. Severe withdrawal (and recovery). *Journal of Philosophical Logic* 28(5):501–547.